Point counting to detect non-permutative elements of $K_1(\mathrm{Var})$

k = finite field

$K_0(\mathrm{Var}_k)$ = Grothendieck ring of varieties

= free ab. gp gen by varieties$/k$ $\Big/$ $Y \overset{cl}{\hookrightarrow} X$, $[X] = [Y] + [X \setminus Y]$

$[X][Y] = [X \times Y]$

Universal additive invariant

$$\{\mathrm{Var}\} \xrightarrow{\mu} A^{\text{ab. gp}}$$

$\mu(X) = \mu(Y) + \mu(X \setminus Y)$

factors uniquely through $K_0(\mathrm{Var}_k)$

ring hom.

$\mu(X \times Y) = \mu(X)\mu(Y)$ multiplicative inv.

Ex: • k finite: point counting

$$X \longmapsto \# X(k)$$

$$K_0(\mathrm{Var}) \longrightarrow \mathbb{Z} = \boxed{K_0(\mathrm{FinSet})}$$

$K_0(\mathrm{FinSet})$ = free ab. gp on finite sets $/ [A \amalg B] = [A] + [B]$

• $k = \mathbb{C}$

mixed Hodge str.

$$X \longmapsto \sum (-1)^i [H^i_c(X)] \in \boxed{K_0(\mathrm{MHS})}$$

$K_0(\mathrm{MHS})$ = free ab. gp on MHS $\{B\}$ $\Big/$ $A \hookrightarrow B \twoheadrightarrow C$, $[B] = [A] + [C]$

• Local zeta function k finite

$$X \longmapsto \sum_{i=0}^{2\dim X} (-1)^i \det\left(1 - t \cdot \mathrm{Frob} \circlearrowleft H^i_c(X; \mathbb{Z}_\ell)\right) \in K_0\left(\begin{smallmatrix}\mathrm{Gal}\\ \mathrm{rep}\end{smallmatrix}\right)$$

$$K_0(\mathrm{Var}_k) \longrightarrow \in (1 + t\mathbb{Z}[[t]], \times) \quad \text{Big Witt ring}$$

$$\cup\!\!\!\!\!\!\;\;|$$

$$X \longmapsto \exp \sum_{n \geq 1} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n \qquad k = \mathbb{F}_q$$

Thm: $\exists$ a spectrum $K(\mathrm{Var})$ s.t. $\pi_0 K(\mathrm{Var}) = K_0(\mathrm{Var})$ and the spectrum is meaningful.

Classical situation: rings

$K_0(R) =$ free ab gp gen by f.g. proj R-mod $\Big/$ $A \hookrightarrow B \twoheadrightarrow C$ exact, $[B] = [A] + [C]$.

Ex:

R field: $K_0(R) = \mathbb{Z}$

R Dedekind domain: $K_0(R) = \mathbb{Z} \oplus Cl(R)$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad [A] + [D] = [B] + [C]$$

$\exists$ spectrum $K(R)$, $K_n(R) := \pi_n K(R)$

$$K_1(R) = GL(R)^{ab} \qquad GL(R) = \operatorname{colim}_n GL_n(R)$$

what does this mean?

One interpretation: $K_1(R)$ is the "natural home" of the determinant.

det:
- homomorphism $GL_n(R) \longrightarrow A$ ← ind. of n
- additive:

$$\begin{tikzcd} GL_n(R) \oplus GL_m(R) \arrow[r, hook] \arrow[d, "\det \oplus \det"'] & GL_{n+m}(R) \arrow[r, "\det"] & A \\ A \oplus A \arrow[urr, "+"'] & & \end{tikzcd}$$

R field: $A = R^\times$, $GL_1 R \xrightarrow{id} R^\times$

$\Rightarrow$ det: $GL(R) \longrightarrow A$ $\overset{?}{=} GL(R)^{ab}$

- $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$\boxed{\text{det: living in } K_1(R)}$

$K_1(\mathrm{Var})$:
- det: $\mathrm{Aut}(X) \longrightarrow A$ ← ind. of X
- additive:

$$\begin{tikzcd} \mathrm{Aut}(X) \oplus \mathrm{Aut}(Y) \arrow[r, hook] \arrow[d, "\det \oplus \det"] & \mathrm{Aut}(X \sqcup Y) \arrow[d, "\det"] \\ A \oplus A \arrow[r, "+"] & A \end{tikzcd}$$

- Computations: how does swapping work?

  does $k^\times$ live in $K_1(\mathrm{Var}_k)$?

"Def": $K_1(\mathrm{Var}) :=$ free ab. gp gen by ? $\boxed{\text{piecewise-automorphisms}}$ of varieties $[X, \tau]$ $\Big/$

group

$[X,\tau] + [Y,\sigma] = [X \sqcup Y, \tau \sqcup \sigma]$

$[X,\tau] + [X,\sigma] = [X, \tau \circ \sigma]$ ← lot of work!

Piecewise-aut: X, two stratifications $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$, $X_0' \subseteq X_1' \subseteq \dots \subseteq X_n' = X$ + isos $X_i \setminus X_{i-1} \xrightarrow{\simeq} X_i' \setminus X_{i-1}'$

Classically: (Quillen)

$\mathcal{E}$ abelian/exact category

$K(\mathcal{E})$ = keep track of which objects are smaller than others

$= \Omega |Q\mathcal{E}|.$

$Q\mathcal{E}$: ob $\mathcal{E}$

mor: formal compositions of $\hookrightarrow$ $\twoheadleftarrow$

composition: a way of commuting these past one another

$\twoheadleftarrow \; \hookrightarrow$

$K_0(R) = \pi_0 K(\mathrm{Mod}_R^{f.g.proj})$

$K_1(R) = \pi_1 K(\mathrm{Mod}_R^{f.g.proj})$

Combinatorial: no additive structure

ex: finite sets, varieties, polyhedra

comes with data about which families "cover" other objects

Def: An (closed) assembler is a Grothendieck site (category) w/ the following extra conditions:

(I) It has an initial object (M) all morphisms are monic (P) has pullbacks.

Ex: Finite sets + injections

$\{f_i: A_i \to A\}_{i\in I}$ cov. fam. if $\bigcup_{i\in I} f_i(A_i) = A$.

Ex: Varieties$/k$ ← locally closed immersions

cov. fam. are gen by $\{Y \overset{cl}{\hookrightarrow} X,\ X\setminus Y \hookrightarrow X\}$ + finite refinements.

Ex: Almost-finite sets: set S w/ $\hat{\mathbb{Z}}$-action s.t. $\forall$ closed subgroup H, S^H is finite.

mor: injections, cov. families $\{f_i: A_i \to A\}_{i\in I}$ if $\bigcup f_i(A_i) = A$.

Ex: $X(\bar{k})$

Thm: $\exists$ a functor K: Assemblers $\longrightarrow$ $\mathcal{S}p$ s.t.

$$K_0(\mathcal{C}) = \text{free ab gr on ob } \mathcal{C} \Big/ \begin{array}{l}\{f_i: A_i \to A\}_{i\in I} \text{ finite disjoint cov. fam} \\ [A] = \sum [A_i]\end{array}$$

I is finite $\quad A_i \times_A A_j = \emptyset,\ i \neq j$

$\Rightarrow$ If you can derive a motivic measure, i.e. write it as a morphism of assemblers, you get motivic measures $K_1(\mathrm{Var}_k) \longrightarrow K_1(?)$.

So you can use derived motivic measures to distinguish determinants.

Also want: If $K_0(\mathcal{C})$ ring and motivic measure is multiplicative then $K(\mathrm{Var}_k) \longrightarrow K(\mathcal{C})$ $\boxed{\text{is a ring map.}}$

Thm: (Campbell) $K(\mathrm{Var}_k)$ is an E_∞-ring spectrum.

Def: There is a monoidal structure on assemblers. Given assemblers $\mathcal{C}, \mathcal{D}$ define $\mathcal{C} \wedge \mathcal{D}$:
$\mathrm{ob}(\mathcal{C} \wedge \mathcal{D}) = (\mathrm{ob}\,\mathcal{C} \setminus \phi) \times (\mathrm{ob}\,\mathcal{D} \setminus \phi) \cup \{\phi\}$

mor: full subcategory of $\mathcal{C} \times \mathcal{D}$

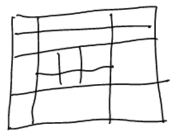
top: Given cov. families $\{A_i \to A\}_{i \in I}^{\in \mathcal{C}}$ $\{B_j \to B\}_{j \in J}^{\in \mathcal{D}}$

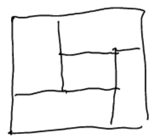
cov. fam: $\{(A_i, B_j) \longrightarrow (A, B)\}_{(i,j) \in I \times J}$ + finite refinements

Ex: $\mathcal{C} = \mathcal{D}$ = closed segments in line of nonzero length + ϕ

$\mathcal{C} \wedge \mathcal{D}$: ob: rectangles in plane $\parallel$ to axes

cov fam: grids + finite refinements





not a cov. fam.

Thm: K takes symmetric monoid objects in <u>Assemblers</u> to E_∞-ring spectra and monoidal morphisms to E_∞-maps.

Ex: Given a variety X, over $\mathbb{F}_q$ zeta function: $\exp \sum_{n \geq 1} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n$ not this

$X(\mathbb{F}_{q^n}) = X(\overline{\mathbb{F}}_q)^{\mathrm{Frob}^n}$ $\Longrightarrow$ $X(\overline{\mathbb{F}}_q)$ + $\hat{\mathbb{Z}}$-action determines the zeta function. look at this

$\mathrm{Var}_k \longrightarrow \mathrm{AFSet}$ this is monoidal:
$X \longmapsto X(\bar{k})$ $\qquad X \times Y \longmapsto (X \times Y)(\bar{k}) = X(\bar{k}) \times Y(\bar{k})$

$\Longrightarrow K(\mathrm{Var}_k) \longrightarrow K(\mathrm{AF\,Set})$ E_∞-map.

Can now try to analyze elements in $K_1(\mathrm{Var})$.

Q: Observation: $\mathrm{FinSet} \longrightarrow \mathrm{Var}_k \longrightarrow \mathrm{Fin\,Set}$ $\Longrightarrow$ $K(\mathrm{FinSet})$ splits off of $K(\mathrm{Var}_k)$.
$S \longmapsto \coprod_S \mathrm{Spec}\,k$
$X \longmapsto X(k)$

Can think of image $K_n(\mathrm{FinSet}) \longrightarrow K_n(\mathrm{Var}_k)$ as "0-dim'l" elts.

<u>Thm</u> (Campbell–Wolfson–Z) $\exists$ non-0-dim'l elts in $K_1(\mathrm{Var}_k)$ when $k = \mathbb{F}_q$, $q \equiv 3\ (4)$.

The ring structure gives a map

$$K_0(\mathrm{Var}) \otimes K_1(\mathrm{FinSet}) \longrightarrow K_1(\mathrm{Var})$$

$[X]$, $\quad K_1(\mathrm{FinSet}) \cong \mathbb{Z}/2$ gen by $\eta = [* \ *, \mathrm{swap}]$

$$[X] \otimes \eta \longmapsto [X \sqcup X, \mathrm{swap}]$$

image: <u>permutative elts</u>

Q: Do there exist non-permutative elts?

<u>Prop</u>: $K(\mathrm{aFSet}) \xrightarrow{\psi_n} K(\mathrm{FinSet}_{\mathbb{Z}/n})$ $\qquad K_1(\mathrm{FinSet}_{\mathbb{Z}/n}) \cong \underset{\text{sign}}{\mathbb{Z}/2} \oplus \underset{\text{internal}}{\mathbb{Z}/n}$

$S \longmapsto$ orbits of size n

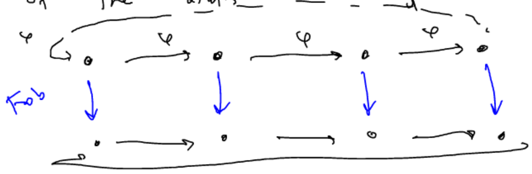
The image of any non-permutative elt in $K_1(\mathrm{Var}_k)$ under $\psi_0 \circ \mathrm{Zeta}$ has internal coordinate equal to 0.

<u>Ex</u>: $k = \mathbb{F}_5$ $\quad X = \{y^2 = x^3 + x\}$ $\quad$ automorphism: $(x,y) \xmapsto{\varphi} (-x, iy)$ $\quad i = 2$

ψ_2: looks at Galois orbits of $X(\overline{\mathbb{F}}_5)$ of size 2: points in $X(\mathbb{F}_{25})$ <u>not</u> in $X(\mathbb{F}_5)$

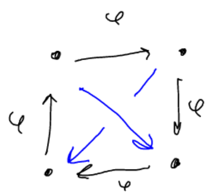
28 of them, in 7 φ-orbits

6 of the orbits are paired up by Galois action



all points are <u>distinct</u> points in $X(\mathbb{F}_{25})$

7th orbit:



$\Longrightarrow \psi_2[X, \varphi] = (-1,-1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$